Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 227 (2008) 1609–1612
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Short Note

Coupling and decoupling of the acoustic and gravity waves through perturbational analysis of the Euler equations

Caroline Gatti-Bono *

Lawrence Livermore National Laboratory, Center for Applied Scientific Computing, 7000 East Avenue L-560, Livermore, CA 94550, United States

Received 3 July 2007; received in revised form 27 September 2007; accepted 29 September 2007
Available online 10 October 2007

Keywords: Non-hydrostatic atmospheric model; Gravity waves

1. Introduction

The fully compressible inviscid models for atmospheric flows contain two types of stiff waves: acoustic waves and gravity waves. Both types of waves impose a time step constraint for explicit numerical models, compared to the background advection of the fluid. In a previous study [2], we presented a splitting and used an asymptotic analysis to show that it lead to the separation of the acoustic and buoyant time scales. To this end, the paper introduced an operator $\mathscr{L}_z$

$$\mathscr{L}_z = -\frac{\partial}{\partial z}\left[\frac{\eta_0 g}{\rho_0 N^2}\left(\frac{1}{g}\frac{\partial}{\partial z}+\frac{1}{c_0^2}\right)\right]+\frac{\eta_0}{\rho_0 c_0^2} \quad (1)$$

where g is the gravity constant, ρ_0 the initial background density, c_0 the initial sound speed, p_0 the initial hydrostatic background pressure and N is the Brunt–Väisälä frequency defined as

$$\frac{N^2}{g} = \frac{1}{\gamma p_0}\frac{\mathrm{d}p_0}{\mathrm{d}z} - \frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} \quad (2)$$

and η_0 is defined as

$$\frac{1}{\eta_0}\frac{\partial \eta_0}{\partial z} = -\frac{\rho_0 g}{\gamma p_0} \quad (3)$$

DOI of original article: 10.1016/j.jcp.2005.12.017
* Tel.: +1 925 424 2084.
E-mail address: cbono@llnl.gov

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2007.09.024

A discussion on the sign of ξ showed that the operator $\mathscr{L}_z$ had the potential of having negative eigenvalues, which had a stabilizing effect since the negative eigenvalues did not support gravity modes. We verified numerically that the simulations were indeed stable for a CFL condition corresponding to the first real eigenvalue, and we showed that this analysis was compatible with the traditional analysis that exists for incompressible or Boussinesq flows. Because this result can seem counter-intuitive, the present study uses perturbational analysis to show that the operator $\mathscr{L}_z$ is intrinsically in the equations and, therefore, is not related to the specifics of the dynamic splitting or the asymptotic analysis of [2]. We follow the methodology traditionally used for incompressible or Boussinesq equations [1] but with the fully compressible Euler equations, and cast the final equation as a function of pressure [3]. The final equation for the total pressure shows two parts: a part containing the acoustic motions and a part containing the gravity wave motions. The latter part is identical to the equation found in [2] for the pressure π_{H} that contained the gravity wave motions (the splitting of [2] allowed to fully decouple this part of the motion from the acoustic motions), and the same operator $\mathscr{L}_z$ appears. This shows that the operator $\mathscr{L}_z$ is not particular to our analysis [2].

2. Perturbational analysis of the Euler equations

Consider the Euler equations for a compressible inviscid fluid in two-dimensions

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \mathbf{grad}(\mathbf{u}) + \frac{1}{\rho}\mathbf{grad}(p) + g\mathbf{k} = 0 \tag{5}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \mathbf{grad}(p) + \rho c^2 \operatorname{div}(\mathbf{u}) = 0 \tag{6}$$

where ρ is the density, $\mathbf{u}$ the velocity and p is the pressure.

We consider the reference state $(\rho = \rho_0(z), \mathbf{u} = 0, p = p_0(z))$ in hydrostatic balance

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g \tag{7}$$

and we linearize around this reference

$$\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\rho_0 \mathbf{u}) = 0 \tag{8}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} = 0 \tag{9}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z} + \frac{\tilde{\rho}}{\rho_0} g = 0 \tag{10}$$

$$\frac{\partial \tilde{p}}{\partial t} + \rho_0 c_0^2 \operatorname{div}(\mathbf{u}) - \rho_0 w g = 0 \tag{11}$$

where $\tilde{\rho} = \rho - \rho_0$ is the perturbational density and $\tilde{p} = p - p_0$ is the perturbational pressure.

Now, we change the primary variables from $(\tilde{\rho}, u, w, \tilde{p})$ to $(\tilde{\theta}, u, w, \tilde{p})$, where $\tilde{\theta}$ is the perturbational potential temperature defined by

$$\tilde{\theta} = \theta - \theta_0(z) \tag{12}$$

$$\theta = \left(\frac{p_{00}^{\frac{\gamma-1}{\gamma}}}{R}\right)\frac{p^{\frac{1}{\gamma}}}{\rho} \tag{13}$$

Using the definitions (12) and (13), we obtain the following relation between the perturbational variables

$$\frac{\tilde{\theta}}{\theta_0} = \frac{1}{\gamma}\frac{\tilde{p}}{p_0} - \frac{\tilde{\rho}}{\rho_0} \tag{14}$$

Combining Eqs. (8) and (11), we obtain an equation for $\tilde{\theta}$ and the full system of equation is

$$\frac{\partial\tilde{\theta}}{\partial t}+\frac{N^2}{g}\theta_0 w=0 \tag{15}$$

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial x}=0 \tag{16}$$

$$\frac{\partial w}{\partial t}+\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial z}+\frac{\tilde{\rho}}{\rho_0}g=0 \tag{17}$$

$$\frac{\partial\tilde{p}}{\partial t}+\rho_0 c_0^2\,\mathrm{div}(\mathbf{u})-\rho_0 wg=0 \tag{18}$$

Now, we want to get an equation only in $\tilde{p}$. We differentiate (18) with respect to t and use Eqs. (16) and (17)

$$\left[\frac{\partial^2\tilde{p}}{\partial t^2}-c_0^2\frac{\partial^2\tilde{p}}{\partial x^2}\right]-\mathcal{M}_z\left[\frac{1}{\rho_0}\mathcal{N}_z\tilde{p}-\frac{g}{\theta_0}\tilde{\theta}\right]=0 \tag{19}$$

where we have defined $\mathcal{M}_z$ and $\mathcal{N}_z$ to be

$$\mathcal{M}_z=\rho_0 c_0^2\left(\frac{\partial}{\partial z}-\frac{g}{c_0^2}\right) \tag{20}$$

$$\mathcal{N}_z=\frac{\partial}{\partial z}+\frac{g}{c_0^2} \tag{21}$$

We differentiate (15) with respect to t and use Eq. (17)

$$\left[\frac{\partial^2\tilde{\theta}}{\partial t^2}+N^2\tilde{\theta}\right]-\frac{N^2\theta_0}{\rho_0 g}\mathcal{N}_z\tilde{p}=0 \tag{22}$$

Combining Eqs. (19) and (22) yields

$$-\mathcal{M}_z\frac{1}{N^2}\mathcal{M}_z^{-1}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2\tilde{p}}{\partial t^2}-c_0^2\frac{\partial^2\tilde{p}}{\partial x^2}\right]+\left(\mathcal{M}_z\frac{1}{\rho_0 N^2}\mathcal{N}_z-\mathcal{I}\right)\frac{\partial^2\tilde{p}}{\partial t^2}+c_0^2\frac{\partial^2\tilde{p}}{\partial x^2}=0 \tag{23}$$

where $\mathcal{I}$ is the identity operator, and finally

$$-\mathcal{M}_z\frac{1}{N^2}\mathcal{M}_z^{-1}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2\tilde{p}}{\partial t^2}-c_0^2\frac{\partial^2\tilde{p}}{\partial x^2}\right]+c_0^2\left[-\left(\frac{\rho_0}{\eta_0}\right)\mathcal{L}_z\frac{\partial^2\tilde{p}}{\partial t^2}+\frac{\partial^2\tilde{p}}{\partial x^2}\right]=0 \tag{24}$$

where $\mathcal{L}_z$ and η_0 were defined in Eqs. (1)–(3).

Eq. (24) is composed of two parts. The first bracket contains the acoustic motions and the second bracket contains the gravity waves motions. We recover the operator $\mathcal{L}_z$ that was introduced in [2], proving that this operator is intrinsic to the fully compressible equations and is not introduced by the particulars of the splitting and asymptotic analysis of [2]. The part of the equation containing the gravity wave motions is identical to the single equation in π_{H}

$$-\left(\frac{\rho_0}{\eta_0}\right)\mathcal{L}_z\frac{\partial^2\pi_H}{\partial t^2}+\frac{\partial^2\pi_H}{\partial x^2}=\mathrm{O}(\varepsilon^2) \tag{25}$$

obtained from the system of hyperbolic equations derived in [2]

$$\mathcal{L}_z\frac{\partial\pi_{\mathrm{H}}}{\partial t}+\eta_0\frac{\partial u_{\mathrm{d}}}{\partial x}=\mathrm{O}(\varepsilon^2) \tag{26}$$

$$\frac{\partial u_{\mathrm{d}}}{\partial t}+\frac{1}{\rho_0}\frac{\partial\pi_{\mathrm{H}}}{\partial x}=\mathrm{O}(\varepsilon^2) \tag{27}$$

where u_{d} is the horizontal solenoidal velocity

$$\mathrm{div}(\eta_0\mathbf{u}_{\mathrm{d}})=0 \tag{28}$$

and π_H is defined as the perturbational hydrostatic pressure that contains the effects of gravity wave motions

$$\frac{\partial \pi_H}{\partial z} = -\tilde{\rho} g \tag{29}$$

3. Conclusion

We have shown that the operator $\mathcal{L}_z$ is structurally in the Euler equations and does not depend on the specifics of the splitting and asymptotic analysis performed in [2]. Therefore, the full compressibility effects have a stabilizing influence for certain stratifications of the atmosphere. The comparison of the structure of Eq. (24) along with Eq. (25) shows that the splitting introduced in [2] fully decouples the acoustic motions from the gravity wave motions.

Acknowledgments

This work was performed under the auspices of the US Department of Energy by the University of California, Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48. UCRL-JRNL-232445.

References

[1] D. Durran, Numerical Methods for Wave Equations in Geophysical Fluid Dynamics, Springer, 1998.
[2] C. Gatti-Bono, P. Colella, An anelastic allspeed projection method for gravitationally stratified flows, Journal of Computational Physics 216 (2) (2006) 589–615.
[3] Thuburn, Normal modes of deep atmospheres. I: Spherical geometry, Quarterly Journal of the Royal Meteorological Society 128 (584) (2002) 1771–1792.